A Fibonacci–Lucas Decomposition of Subdivision Lengths in the Pentagram

Samuel H. Larsen
sam@slarsen.io

July 9, 2025

Abstract

We investigate the subdivision hierarchy produced by the step-2 chords of a regular star polygon $\{n/2\}$. For the pentagram $\{5/2\}$ we show that every segment length is a power of $\lambda = \phi^{-1}$. Writing $\phi^{-k} = a_k + b_k\sqrt{5}$ for $k \in \{0, 1, 2, 3\}$, we prove $a_k = (-1)^k L_k/2$ and $b_k = (-1)^{k+1} F_k/2$, where F_k and L_k are the Fibonacci and Lucas numbers. The coefficient $|b_k|$ reaches the value 1 for the first time at $k = 3$; remarkably, $k = 3$ is also the final possible layer, because each step-2 chord is intersected at most twice. Thus the algebraic unit-coefficient threshold and the geometric endpoint of subdivision coincide only in the pentagram, highlighting its special role among $\{n/2\}$ stars and providing a benchmark for extensions to other stellations.

1 Introduction

Regular star polygons reveal a subtle interplay between elementary number theory and planar geometry. The most familiar example, the pentagram, scales by the reciprocal of the golden ratio ϕ, so that its visible segment lengths form the cascade $1, \phi^{-1}, \phi^{-2}, \phi^{-3}$. Writing each power as $a_k + b_k\sqrt{5}$ exposes a Fibonacci–Lucas pattern in the coefficients, prompting the question: at which subdivision layer does the irrational $\sqrt{5}$ term first appear with full unit weight?

Theorem 1 answers that question, proving that the jump occurs at layer $k = 3$ and never sooner. Remark 1 shows that no further layers can arise in any star polygon $\{n/2\}$, because each step-2 chord is crossed at most twice. Hence, only in the pentagram does the algebraic unit-coefficient threshold coincide with the geometric endpoint of the subdivision process. This coincidence places $\{5/2\}$ in a unique position among regular star polygons and sets a benchmark for extending Fibonacci–Lucas phenomena to broader classes of stellations, such as those arising in stellated polyhedra.

2 Definitions

Definition 1 (Star polygon $\{n/2\}$)**.** *Let $n \geq 5$ be an integer. The* star polygon *with Schläfli symbol $\{n/2\}$ is obtained by joining each of n equally spaced vertices on a circle to the vertex two steps ahead. If n is odd, the result is an equiangular, equilateral, self-intersecting, unicursal regular star polygon; if n is even, it is the regular compound of two congruent $(n/2)$-gons (e.g. the hexagram when $n = 6$).*

Definition 2 (Golden ratio)**.** *The* golden ratio*, denoted ϕ, is the unique positive real number that satisfies*

$$\phi^2 \;=\; \phi + 1, \qquad \phi > 0.$$

Solving the quadratic gives

$$\phi \;=\; \frac{1+\sqrt{5}}{2} \;\approx\; 1.618\,034.$$

Equivalently, when a line segment of total length $a + b$ is divided into parts with lengths $a > b > 0$, the common value of the two ratios is the golden ratio:

$$\phi \;=\; \frac{a+b}{a} \;=\; \frac{a}{b}.$$

The reciprocal of the golden ratio is

$$\phi^{-1} \;=\; \frac{\sqrt{5}-1}{2} \;=\; \phi - 1 \;\approx\; 0.618\,034.$$

Definition 3 (Subdivision layers)**.** *Let λ be the characteristic scaling ratio of a star polygon $\{n/2\}$. The* subdivision layers *are indexed by integers $k \in \{0, 1, 2, 3\}$, with layer 0 the outermost star edges. Layer k has characteristic edge lengths scaled by λ^k relative to those of layer 0.*

Definition 4 (Pentagram)**.** *The* pentagram *is the regular star polygon with Schläfli symbol $\{5/2\}$, obtained by joining each of five equally spaced vertices on a circle to the vertex two steps ahead; it is a self-intersecting, unicursal, equiangular, and equilateral polygon. When the outer edges of the pentagram have length 1 we speak of a* unit pentagram*. In this normalization every segment length is a power of the similarity factor*

$$\lambda = \phi^{-1} = \tfrac{\sqrt{5}-1}{2} \approx 0.618,$$

namely λ^k with $k \in \{0, 1, 2, 3\}$.

3 Preliminaries

Lemma 1 (Fibonacci coefficients in pentagram subdivision layers)**.** *Let $\lambda = \phi^{-1}$ be the characteristic scaling ratio of the pentagram. For every $k \in \mathbb{N}_0$,*

$$\phi^{-k} = a_k + b_k\sqrt{5}, \qquad b_k = (-1)^{k+1}\frac{F_k}{2}, \qquad a_k = (-1)^k\frac{L_k}{2},$$

where F_k and L_k are the k-th Fibonacci and Lucas numbers, respectively ($F_0 = 0, F_1 = 1;\ L_0 = 2, L_1 = 1$). Hence $|b_k| = |F_k|/2$; the magnitude of the $\sqrt{5}$-term at subdivision layer k is governed by the Fibonacci sequence. Explicitly,

$$\begin{aligned}
&\textit{Layer } 0\ (\lambda^0): && F_0 = 0, && b_0 = 0, && \phi^0 = 1,\\
&\textit{Layer } 1\ (\lambda^1): && F_1 = 1, && b_1 = \tfrac{1}{2}, && \phi^{-1} = \tfrac{\sqrt{5}-1}{2} \approx 0.618,\\
&\textit{Layer } 2\ (\lambda^2): && F_2 = 1, && b_2 = -\tfrac{1}{2}, && \phi^{-2} = \tfrac{3-\sqrt{5}}{2} \approx 0.382,\\
&\textit{Layer } 3\ (\lambda^3): && F_3 = 2, && b_3 = 1, && \phi^{-3} = \sqrt{5} - 2 \approx 0.236.
\end{aligned}$$

The first layer with $|b_k| = 1$ is $k = 3$, marking the appearance of an irrational $\sqrt{5}$-term whose coefficient has unit magnitude in the subdivision hierarchy. In a star polygon $\{n/2\}$ this is also the smallest (and final) subdivision layer (see Remark 1).

Remark 1. *In any star polygon $\{n/2\}$ the subdivision process stops after three layers: each step-2 chord is crossed by at most two other chords, so no new intersection points arise beyond layer $k = 3$.*

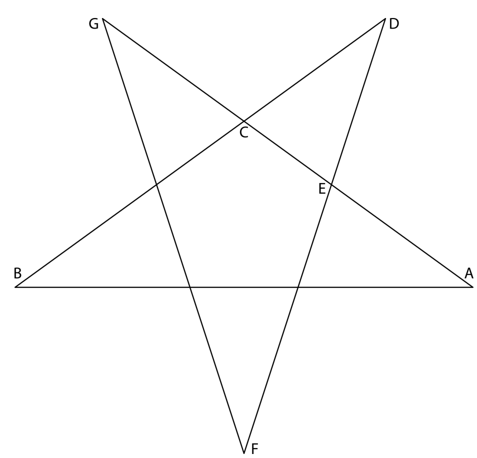

Figure 1: Construction of a regular pentagram showing vertices A, B, D, F, G and intersection points C, E.

Lemma 2 (Powers of ϕ^{-1} in the unit pentagram)**.** *In the* unit pentagram *(each outer edge has length* 1*) every segment created by the intersections of step-2 chords has length ϕ^{-k} for some $k \in \{0, 1, 2, 3\}$. Equivalently, the set of distinct lengths is $\{1, \phi^{-1}, \phi^{-2}, \phi^{-3}\}$.*

Proof. Normalize the pentagram in Figure 1 so that all five outer edges have length $\lambda^0 = 1$

$$|AB| = 1.$$

An interior intersection point C marking length $\lambda^1 = \phi^{-1}$

$$|AC| = \tfrac{\sqrt{5}-1}{2} \approx 0.618.$$

An outer segment marking length $\lambda^2 = \phi^{-2}$

$$|AE| = \tfrac{3-\sqrt{5}}{2} \approx 0.382.$$

An inner segment marking length $\lambda^3 = \phi^{-3}$

$$|CE| = \sqrt{5} - 2 \approx 0.236.$$

Each step-2 chord is intersected at most twice (Remark 1), so every segment produced anywhere in the pentagram is congruent to one of the four lengths $\{\lambda^0, \lambda^1, \lambda^2, \lambda^3\}$. Thus all unit pentagram subdivision layers are powers of ϕ^{-1}. □

Remark 2. *In the unit pentagram the triangle $\triangle ABC$ consists of one layer-0 edge $|AB| = 1$ and two layer-1 edges $|AC| = |BC| = \phi^{-1}$. Its perimeter is*

$$P(\triangle ABC) = |AB| + 2|AC| = 1 + 2\phi^{-1} = \sqrt{5}.$$

Hence a length $\sqrt{5}$ emerges from the subdivision layers in the unit pentagram.

4 Main Results

Theorem 1 (First unit-coefficient layer at minimum depth in the unit pentagram)**.** *Let $\lambda = \phi^{-1}$ be the characteristic scaling ratio of the unit pentagram $\{5/2\}$, and write*

$$\phi^{-k} = a_k + b_k\sqrt{5}, \qquad k \in \{0, 1, 2, 3\},$$

with $a_k, b_k \in \mathbb{Z}[\tfrac{1}{2}]$ as in Lemma 1*. Then*

$$|b_0| = |b_1| = |b_2| < 1 \quad \textit{and} \quad |b_3| = 1.$$

Therefore

1. *the first layer in which an irrational $\sqrt{5}$–term appears with* unit *coefficient is $k = 3$; and*

2. *by* Remark 1 *there are no further subdivision layers, so* $k = 3$ *is simultaneously the first unit-coefficient layer* and *the last possible layer in the* $\{n/2\}$ *subdivision hierarchy.*

Proof. Lemma 1 gives $b_k = (-1)^{k+1}F_k/2$. Because $F_0 = 0$ and $F_1 = F_2 = 1$, we have $|b_0| = 0$ and $|b_1| = |b_2| = \frac{1}{2} < 1$. For $k = 3$ one has $F_3 = 2$, so $|b_3| = 1$. No larger k occurs in a $\{n/2\}$ star (Remark 1), so the inequalities above exhaust all possible layers. □

5 Conclusion

We have given a self-contained algebraic–geometric analysis of the regular star polygon family $\{n/2\}$ with special attention to the pentagram $\{5/2\}$.

5.1 Algebraic and Geometric Analysis

Lemma 1 decomposes every scaling power ϕ^{-k} ($k \in \{0, 1, 2, 3\}$) into half-integer coefficients $a_k + b_k\sqrt{5}$ with $b_k = (-1)^{k+1}F_k/2$ and F_k the Fibonacci sequence. The table shows that $|b_k|$ climbs from 0 to $\frac{1}{2}$ and reaches 1 exactly at $k = 3$.

Lemma 2 proves that in the unit pentagram every segment length is one of the four $\lambda^0, \lambda^1, \lambda^2, \lambda^3$ with $\lambda = \phi^{-1}$; Remark 1 explains why no further subdivision layers can occur—each step-2 chord is intersected at most twice.

5.2 Synthesis

Theorem 1 identifies the first unit-coefficient layer at minimum depth in the unit pentagram, showing that layer $k = 3$ is simultaneously:

- the *first* layer whose scaling factor carries an irrational $\sqrt{5}$-term with unit coefficient—something that never occurs in any other star polygon $\{n/2\}$ with $n \neq 5$; and
- the *last* layer that can appear in any $\{n/2\}$ subdivision hierarchy, because each step-2 chord is intersected at most twice.

Thus the algebraic unit-coefficient threshold and the geometric endpoint of the subdivision process coincide only for the pentagram.

5.3 Outlook

Two directions invite further study. One is algebraic: extend the Fibonacci–Lucas decomposition to broader families of star polygons $\{n/k\}$ with $k \neq 2$ or to higher-order chords inside the same $\{n/2\}$ figures. The other is geometric: seek three-dimensional analogues in stellated polyhedra, where similar scale-factor cascades may expose new links between number theory and polyhedral geometry. The layer-3 coincidence established in this work provides a precise benchmark for any such generalization.